Synchronization in Nonlinear Systems Under Multiplicative Perturbations Over its Linear Parts

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Abstract. The objective of this paper is to show that under a defined multiplicative group, hyperbolic equilibrium points of a dynamical system are preserved along the stable and unstable manifolds. As a consequence, synchronization is also preserved for a master/slave system configuration. The properties of this multiplicative group are determined through the use of simultaneous Schur decomposition.

Key words: Control, Preservation of Synchronization, Chaotic systems, Nonlinear systems, Output feedback and Observers

1 Introduction

The study of synchronization preservation is relevant when it comes to chaos control problems. As a matter of fact, the generalized synchronization can even be derived for different systems by finding a diffeomorphic transformation such that the states of the slave system can be written as a function of the states of the master dynamics (see [1] and references therein). This result can be seen as a timely contribution; however, in accordance to the goal of keeping intact the stability under the transformation, a new question arises: how can stability be preserved under transformations suffered by a dynamical system? An answer to this question might allow us to ensure synchronization in strictly different systems, in the sense that stability of the error dynamics is preserved under the transformation. Preservation of stability for a class of nonlinear autonomous dynamical systems has been reported in the last decades [2]. The underlying idea is to preserve the stability properties under transformation of finite-dimensional dynamical systems. In the case of linear dynamical systems there exist several results of stability preservation, for instance in [3–5], stability is asymptotically preserved using transformations on rational functions in the frequency domain. We present a simple extension of the Stable-Unstable Manifold Theorem, based on the preservation of the signature of the real parts of the eigenvalues of an underlying Jacobian matrix. The developed methodology is based on the use of matrix theory tools, specifically, simultaneous Schur decomposition, the multiplicative group structure for triangular and diagonal matrices, the closure under product of positive definite matrices and the eigenvalue sign-preservation for

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both real and complex matrices under matrix multiplication. The results on preservation of structure, stability and synchronization based on the extension of the stable-unstable manifold theorem show that stability and synchronization can be preserved by transforming the linear part of the synchronization system.

2 Structure Preservation

In this section we present the necessary definitions and results that will allow us to prove the main propositions of this paper. The results will be used in section 4 where we will present some examples on preservation of synchronization in dynamical systems.

Simultaneous Schur decompositions are defined as follows

Definition 1. *The group of matrices $A_1, A_2, \ldots, A_n$ is said to be Schur simultaneously decomposable if there exists a unitary matrix U, where $UU^\top = U^\top U = I$, such that*

$$A_1 = UT_1U^\top,\ A_2 = UT_2U^\top,\ \ldots,\ A_n = UT_nU^\top$$

where T_i is an upper triangular matrix.

For the following discussion consider the dynamical system described by

$$\dot{x} = f(x)$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous differentiable function of its argument. Let $A = \left.\frac{\partial f}{\partial x}\right|_{x_0}$ be the Jacobian matrix associated with f evaluated at an equilibrium point x_0.

We introduce the following lemma in order to establish structure preservation under matrix multiplication of simultaneous Schur decomposable matrices.

Lemma 1. *The modifying matrix M, which is a simultaneous block diagonal Schur decomposition of our dynamical system's associated Jacobian matrix A, will maintain the system's structure.*

Proof. We propose a matrix M that is a block diagonal simultaneous Schur decomposition to our matrix A

$$A = UT_AU^\top$$
$$M = UD_MU^\top$$

then our modified system will now become

$$MA = UD_MU^\top UT_AU^\top = UD_MT_AU^\top = U\hat{T}_AU^\top$$

Thus preserving the system's structure.

This small proof enables us to pursue the definition of a multiplicative group under the product of simultaneous Schur decomposable matrices. What follows now is the explicit definition of the multiplicative group and its properties which allow for preservation of stability. Using this group we present an extension of the Stable-Unstable Manifold theorem (Proposition 1).

3 Local Stable-Unstable Theorem Extension

The following proposition is a simple extension of the Local Stable-Unstable Manifold Theorem for the action of group Γ_U on the matrix A and the vector field $f(x)$ where A, the system's linear coefficients matrix, may be decomposed as $A = UT_AU^\top$, with T_A an upper triangular matrix and $UU^\top = U^\top U = I$. We define Δ_{pd} as the set of block diagonal matrices whose real coefficients are all positive and the group Γ_U as

$$\Gamma_U = \left\{ M \in \mathbb{R}^{n\times n} \left| \begin{array}{l} M = UD_MU^\top \\ \text{with } D_M \text{ a block diagonal matrix} \\ \text{and } D_M \in \Delta_{pd} \end{array} \right. \right\}$$

This proposition is an alternative result to proposition 4.2 presented in [6] using simultaneous Schur decomposition and standard matrix product.

Proposition 1. *Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system $\dot{x} = f(x) = Ax + g(x)$. Suppose that $f(0) = 0$ and that $A = Df(0)$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part, i.e., the origin is an hyperbolic fixed point. Then for each matrix $M \in \Gamma_U$, as previously defined, there exists a k-dimensional differentiable manifold S_M tangent to the stable subspace E^S_M of the linear system $\dot{x} = MAx$ at 0 such that for all $t \geq 0$, $\phi_{M,t}(S_M) \subset S_M$ and for all $x_0 \in S_M$,*

$$\lim_{t\to\infty} \phi_{M,t}(x_0) = 0,$$

where $\phi_{M,t}$ be the flow of the nonlinear system $\dot{x} = MAx + g(x)$; and there exists an $n-k$ dimensional differentiable manifold W_M tangent to the unstable subspace E^W_M of $\dot{x} = MAx$ at 0 such that for all $t \leq 0$, $\phi_{M,t}(W_M) \subset W_M$ and for all $x_0 \in W_M$,

$$\lim_{t\to-\infty} \phi_{M,t}(x_0) = 0.$$

An interesting property is that Proposition 1 is valid for each $\overline{g} \in C^1(E)$ such that $\dot{x} = \overline{f}(x) = Ax + \overline{g}(x)$ and

$$\frac{\|\overline{g}(x)\|_2}{\|x\|_2} \to 0 \text{ as } \|x\|_2 \to 0.$$

In consequence, the set of matrices Γ_U generate the action of the group Γ_U on the set of hyperbolic nonlinear systems (formally on the set of hyperbolic vector fields $f \in C^1(E)$) $\dot{x} = \overline{f}(x) = Ax + \overline{g}(x)$ with $\overline{g} \in C^1(E)$ and

$$\Lambda \equiv \left\{ A \in \mathbb{R}^{n\times n} \left| \begin{array}{l} A = UT_AU^\top \text{ with } T_A \\ \text{the Schur decomposition of } A \end{array} \right. \right\}$$

satisfying the last condition, where U is a unitary matrix, this action is faithful and free. The former action is generated by the action of the group Γ_U on the set

A. The action preserves hyperbolic points in nonlinear systems and dimension of the stable and unstable manifolds, i.e, an hyperbolic nonlinear system ($\dot{x} = Ax + \overline{g}(x)$) is mapped in an hyperbolic nonlinear system ($\dot{x} = MAx + \overline{g}(x)$), and $\dim S = \dim S_M$ *and* $\dim W = \dim W_M$.

The following remarks are necessary for our proof

Remark 1. For a complex eigenvalue $a_k + ib_k$, belonging to A where $a_k < 0$; it is required that the eigenvalues of our complex modifying matrix M, $\bar{a}_k + i\bar{b}_k$, fulfill $\bar{a}_k > 0$ and $\pm b_k \Rightarrow \pm\bar{b}_k$.

Remark 2. For a complex eigenvalue $a_k + ib_k$, belonging to A where $a_k > 0$; it is required that the eigenvalues of our complex modifying matrix M, $\bar{a}_k + i\bar{b}_k$, fulfill $\bar{a}_k > 0$ and $\pm b_k \Rightarrow \mp\bar{b}_k$.

Sketch of Proof:

1. Real Coefficients
Consider a matrix A of order $n \times n$ with decomposition $A = UT_AU^\top$, where T_A is an upper triangular matrix with k negative real eigenvalues and $n-k$ positive real eigenvalues and U is a unitary $n \times n$ matrix, and the decomposition $M = UD_MU^\top$, $M \in \Gamma_U$. By our proposition we carry out the matrix product $MA = UD_MU^\top UT_AU^\top = UD_MT_AU^\top$. Looking at the product of triangular and diagonal matrices D_M and T_A whose eigenvalues are $\sigma_M = \{\lambda_i\}$ and $\sigma_A = \{\mu_i\}$, respectively. It is simple to observe that the resulting matrix's eigenvalues are precisely the individual products of the original matrices' eigenvalues; $\sigma_{MA} = \{\lambda_i\mu_i\}$. Since $M \in \Gamma_U$ is strictly positive, then the matrix MA has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Since the dimensions of each manifold have not changed, the result is a consequence of the Stable-Unstable Manifold Theorem and Lemma 1.
2. Complex Coefficients
Consider the same conditions established in the previous point except that the Jacobian and modifying matrices now have complex eigenvalues. Seeing that the product of matrices D_M and T_A results in a block matrix of complex eigenvalues we must adhere to what was previously established in remarks 1 and 2. For the case of the eigenvalues related to matrix A with negative real part $a_k < 0$ (remark 1) we have the following product of eigenvalues $(a_k \pm ib_k)(\bar{a}_k \pm i\bar{b}_k) = a_k\bar{a}_k - \bar{b}_kb_k \pm i(a_k\bar{b}_k + \bar{a}_kb_k)$, since $\bar{a}_k > 0$ it follows that $a_k\bar{a}_k < 0$ and subtracting $\bar{b}_kb_k$ will keep the real part of the new eigenvalue negative. Similarly in the case that $a_k > 0$ (remark 2) we have $(a_k \pm ib_k)(\bar{a}_k \mp i\bar{b}_k) = a_k\bar{a}_k + \bar{b}_kb_k \pm i(a_k\bar{b}_k \mp \bar{a}_kb_k)$, since $\bar{a}_k > 0$ it follows that $a_k\bar{a}_k > 0$ and adding $\bar{b}_kb_k$ will keep the real part of the new eigenvalue positive. Then the matrix MA has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Again the dimension of each manifold is the same as the original system's, therefore the result is again a consequence of the Stable-Unstable Manifold Theorem and Lemma 1.

The relevance of this proposition, which differs from that which appears in [6], resides on the fact that critical hyperbolic equilibrium points are preserved. As a consequence of this we established properties which allow us to preserve the signature of the associated Jacobian matrix.

In sections 4 and 5 it will be shown, through example, that stability and synchronization are preserved under modifications performed on dynamical systems following the methodology of Proposition 1. This proposition on the extension of the Stable-Unstable Manifold Theorem is different to other approaches for stability and synchronization preservation such as [7] where Lyapunov's indirect method was employed.

Notice that given a particular nonlinear system, the stable and unstable manifolds S and W are unique.

4 Preservation of Synchronization in Modified Systems

In this section we show how it is possible to preserve synchronization after a system's eigenvalues have been modified under the action of a class of transformation on the linear part of the nonlinear system.

Consider following n-dimensional systems in a master-slave configuration, where the master system is given by

$$\dot{x} = Ax + g(x)$$

and the slave system is

$$\dot{y} = Ay + f(y) + u(t)$$

where $A \in R^{n \times n}$ is a constant matrix, $f,\ g : R^n \rightarrow R^n$ are continuous nonlinear functions and $u \in R^n$ is the control input. The problem of synchronization considered in this section is the complete-state exact synchronization. That is, the master system and the slave system are synchronized by designing an appropriate nonlinear state feedback control $u(t)$ which is attached to the slave system such that

$$\lim_{t \rightarrow \infty} \|y(t) - x(t)\| \rightarrow 0$$

where $\|\cdot\|$ is the Euclidean norm of a vector.

Considering the error state vector $e = y - x \in R^n$, $f(y) - g(x) = L(x, y)$ and an error dynamics equation

$$\dot{e} = Ae + L(x, y) + u(t).$$

Based in the active control approach [8], to eliminate the nonlinear part of the error dynamics, and choosing $u(t) = Bv(t) - L(x, y)$, where B is a constant gain matrix which is selected such that (A, B) be controllable, we obtain

$$\dot{e} = Ae + Bv(t).$$

Notice that the original synchronization problem is equivalent to the problem of stabilizing the zero-input solution of the last system by a suitable choice of the state feedback control.

Since the pair (A, B) is controllable one such suitable choice for state feedback is a linear-quadratic state-feedback regulator [9], which minimizes the quadratic cost function

$$J(u(t)) = \int_0^\infty \left(e(t)^\top Q e(t) + v(t)^\top R v(t)\right) dt$$

where Q and R are positive semi-definite and a positive definite weighting matrices, respectively. The state-feedback law is given by $v = -Ke$ with $K = R^{-1}B^\top S$ and S the solution to the Riccati equation

$$A^\top S + SA - SBR^{-1}B^\top + Q = 0.$$

This state-feedback law renders the error equation to $\dot{e} = (A - BK)e$, with $(A - BK)$ a Hurwitz matrix[1]. The linear quadratic regulator (LQR) is a well-known design technique that provides practical feedback gains [9]. An interesting property of (LQR) is robustness.

Now consider $M \in \Gamma_U$, and suppose that the following two n-dimensional systems are chaotic

$$\dot{x} = (MA)\, x + g(x)$$
$$\dot{y} = (MA)\, y + f(y) + \widehat{u}(t)$$

for some f, $g : R^n \to R^n$ continuous nonlinear functions and $\widehat{u} \in R^n$ is the control input. We have that $\widehat{u}(t) = -(MBK)\, e - L(x, y)$ stabilizes the zero solution of the error dynamics system, the resultant system

$$\dot{e} = (M\,(A - BK))\, e$$

is asymptotically stable. Notice that using $K = R^{-1}B^\top S$, we obtain

$$\dot{e} = \left(M\left(A - BR^{-1}B^\top S\right)\right) e$$

The original control $u(t) = -BKe - L(x, y)$ is preserved in its linear part by the matrix product $M\,(\cdot)$ and the new control is given by $\widehat{u}(t) = -(MBK)\, e - L(x, y)$. Therefore, we can interpret the last procedure as one in which the controller $u(t)$ which achieves the synchronization in the two original systems is preserved under the transformation $M\,(\cdot)$ so that $\widehat{u}(t)$ achieves the synchronization in the two resultant systems after the transformation. A similar procedure is possible if we consider the transformation $(\cdot)\, M$.

In general, under the transformations $(A, g) \to (MA, \overline{g})$, and under the hypothesis of the existence of a constant state feedback $\mathbf{U} = -Kx$ which achieves synchronization of the original chaotic systems, and also that the transformed systems are chaotic, synchronization can be preserved.

The main contribution in this section does not deal with a better synchronization methodology, rather it deals with the fact that synchronization is preserved when the underlying nonlinear chaotic dynamical system is altered in such a way as to change its dynamical behavior yet preserving the topological structure near the origin.

[1] A Hurwitz matrix is a matrix for which all its eigenvalues are such that their real part is strictly less than zero

5 Synchronization of a Chaotic Attractor

5.1 The Sprott O Attractor

The dynamical system of what is know as the Sprott O attractor, which has chaotic behavior, is defined by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_1 - x_3\\ \dot{x}_3 &= x_1 + 2.7x_2 + x_1x_3\end{aligned}$$

In order to observe synchronization behavior we present two Sprott O attractors arranged as a master/slave configuration. The master and the slave systems are almost identical, the only difference being that the slave system includes an extra term (the control) which is used for the purpose of synchronization with the master system. The initial conditions for the two systems are different.

Considering the errors $e_1 = y_1 - x_1$, $e_2 = y_2 - x_2$, $e_3 = y_3 - x_3$, then the error dynamics equations may be written as

$$\begin{aligned}\dot{e}_1 &= e_2 + u_1(t)\\ \dot{e}_2 &= e_1 - e_3 + u_2(t)\\ \dot{e}_3 &= e_1 + 2.7e_2 + y_1y_3 - x_1x_3 + u_3(t)\end{aligned}$$

Introducing the Jacobian (A) and non-linear terms (L) matrices

$$A = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & -1\\ 1 & 2.7 & 0\end{pmatrix},\ L(x,y) = \begin{pmatrix} 0\\ 0\\ y_1y_3 - x_1x_3\end{pmatrix},\ u = \begin{pmatrix} u_1(t)\\ u_2(t)\\ u_3(t)\end{pmatrix}$$

and selecting the matrix B such that (A, B) is controllable: $B = I$. Now the LQR controller is obtained by using weighting matrices $Q = I$ and $R = B^{\mathsf{T}}B = I$. The state feedback matrix is given by

$$K = \begin{pmatrix} 1.3465 & 0.9819 & -0.0894\\ 0.9819 & 1.7009 & 0.1709\\ -0.0894 & 0.1709 & 0.7881\end{pmatrix}$$

In Fig. 1 the trajectories for the solution of the master system and slave system are shown. In Fig. 2 the absolute value for the errors between the master and slave systems are shown in a semi-logarithmic plot to emphasize the fact that the error converges to zero and therefore the synchronization between the master and slave systems is successfully achieved.

5.2 The Modified Sprott O Attractor

The following example shows the modifications performed on the Sprott O attractor with a complex eigenvalue block diagonal matrix. The general equation

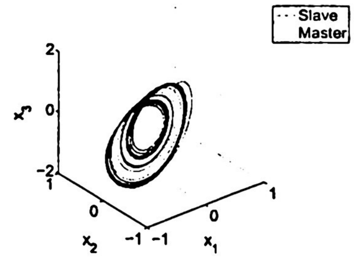

Fig. 1. Original Sprott O attractor showing synchronization between master and slave systems (initial conditions $x_1 = 0.1, x_2 = 0.3, x_3 = 0.2$ and $y_1 = 0.4, y_2 = 0.2, y_3 = 0.1$ respectively).

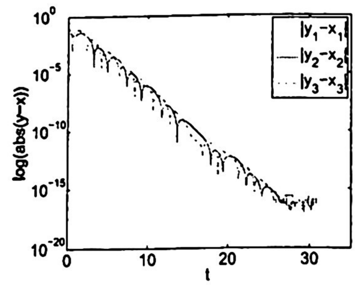

Fig. 2. Magnitude of error $|e| = |y - x|$ between master and slave systems.

for the modified Sprott master and slave systems' linear and non-linear parts may be defined as follows

$$\dot{x} = (MA)x + \begin{bmatrix} 0 \; 0 \; x_1x_3 \end{bmatrix}^{\mathsf{T}},$$
$$\dot{y} = (MA)y + \begin{bmatrix} 0 \; 0 \; y_1y_3 \end{bmatrix}^{\mathsf{T}} + u(t)$$

Considering the error vector $e = y - x$, then the error dynamics may be written as

$$\dot{e} = (MA)e + L(x, y) + u(t)$$

with $u = -L(x, y) + v$ and $v = -(MBK)e$

Defining the modifying matrix

$$M_1 = \begin{pmatrix} 0.9777 & -0.0114 & 0.0224 \\ -0.0169 & 0.9572 & -0.0067 \\ 0.0186 & -0.0142 & 0.9651 \end{pmatrix}$$

which was constructed using simultaneous Schur decomposition and following the sign relationships established in Remark 2 (Since the real coefficients of the complex eigenvalues of A are positive). Once again we use K as in section 5.1 and $u = -(M_1BK)e - L(x, y)$.

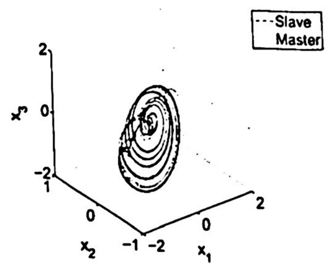

Fig. 3. Master and slave system (initial conditions $x_1 = 0.5, x_2 = 0.03, x_3 = 0.5$ and $y_1 = 0.05, y_2 = 0.03, y_3 = 0.02$ respectively) synchronization of modified Sprott O attractor (with $M_1 \in \Omega_P$ having both real and complex eigenvalues).

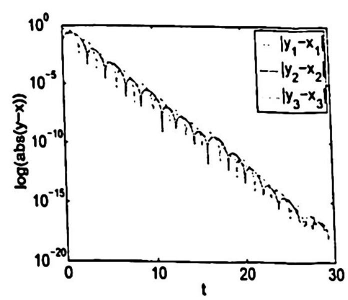

Fig. 4. Magnitude of error $|e| = |y - x|$ between master and slave systems (real and complex eigenvalue modifications).

In Fig. 4 we have the absolute error of the master/slave system configuration, with complex matrix perturbation, and again we see that there is an effective convergence to zero.

Looking at the modified attractor in Fig. 3, as far as we can see, the chaotic dynamics are preserved.

Taking into account that the control input that was applied to achieve synchronization in the presented example was generated through the same methodology used to modify the linear part of the system and that synchronization was achieved, we may infer robustness under multiplicative perturbations over the linear part of nonlinear systems.

6 Conclusions

The preservation of hyperbolic behavior in chaotic synchronization is studied from an extension of the local stable-unstable manifold theorem based in the preservation of the signature of the linear part of the vector fields in nonlinear dynamical systems. It has been shown that a master/slave pair for which synchronization is achieved via the use of a state feedback obtained using a linear-quadratic regulator, synchronization may be preserved even after the master/slave/controller system is transformed. From the results we may conclude that the fundamental properties of the synchronization manifold, the signature of the Jacobian Matrix, hyperbolic equilibrium points and the stability of the system are preserved thus showing that robustness is a consequence of this methodology. The results can be extended to other techniques for the use of feedback design.

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